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# TENSOR PRODUCTS OF APPROXIMATELY COHEN–MACAULAY RINGS

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***Our aim in this article is to study a problem originally raised by Grothendieck. We show that the approximately Cohen–Macaulay property is preserved for the tensor product of algebras over a field k. We also discuss the converse problem.***

***Key Words:*** Approximately Cohen–Macaulay ring; Flat homomorphism of rings; Tensor product.

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## 1. INTRODUCTION

All rings and algebras considered in this article are commutative Noetherian with identity element, and all ring homomorphisms are unital. Throughout, $k$ stands for a field. Let $(R, \mathfrak{m})$ be a local ring with $\dim(R) = d$. Recall that $R$ is a Gorenstein ring if and only if there is an element $a$ of $\mathfrak{m}$ such that $R/a^nR$ is a Gorenstein ring of dimension $d-1$ for every integer $n > 0$ (cf. Hochster, 1977). Clearly, this is not true for Cohen–Macaulay rings. The local ring $R$ is called an *approximately Cohen–Macaulay* ring if either $\dim(R) = 0$ or there exists an element $a$ of $\mathfrak{m}$ such that $R/a^nR$ is a Cohen–Macaulay ring of dimension $d-1$ for every integer $n > 0$ (cf. Goto, 1982). It is shown that if $R$ is an approximately Cohen–Macaulay ring, then so is the ring $R_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ (see Theorem 2). Therefore, the

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concept of approximately Cohen–Macaulay is extended to nonlocal rings as follows. A ring $R$ is an *approximately Cohen–Macaulay* ring if for all prime ideals $\mathfrak{p}$ of $R$, the ring $R_{\mathfrak{p}}$ is an approximately Cohen–Macaulay ring. It is well known that the tensor product $R \otimes_A S$ of regular rings is not regular in general, even if we assume $R$ and $S$ are $A$-algebras, where $A$ is a field (cf. Tousi and Yassemi, 2003, Remark 7). In Watanabe et al. (1969, Remark 1.7), the authors showed that under a suitable condition, tensor products of regular rings are complete intersections. It is proven in Grothendieck (1965) that the tensor product $R \otimes_A S$ of Cohen–Macaulay rings are again Cohen–Macaulay, if we assume that $R$ is a flat $A$-module and $S$ is a finitely generated $A$-module, and in Watanabe et al. (1969), it is shown that the same is true for Gorenstein rings. In Bouchiba and Kabbaj (2002), the authors showed that if $R$ and $S$ are $k$-algebras such that $R \otimes_k S$ is Noetherian, then $R \otimes_k S$ is a Cohen–Macaulay ring if and only if $R$ and $S$ are Cohen–Macaulay rings. Recently, Tousi and Yassemi (2003) showed that if $R$ and $S$ are nonzero $k$-algebras such that $R \otimes_k S$ is Noetherian, then $R \otimes_k S$ is a locally complete intersection (resp. Gorenstein, Cohen–Macaulay) if and only if $R$ and $S$ are locally complete intersections (resp. Gorenstein, Cohen–Macaulay).

In this article we shall investigate if the approximately Cohen–Macaulay property is conserved under tensor product operations. It is shown that if $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ is a flat local homomorphism and $R$ is not a Cohen–Macaulay ring, then the following are equivalent (see Theorem 6):

(a) $R$ is an approximately Cohen–Macaulay ring and $S/\mathfrak{m}S$ is a Cohen–Macaulay ring;
(b) $S$ is an approximately Cohen–Macaulay ring and $\mathrm{Ass}_S(S/\mathfrak{p}S) = \mathrm{Assh}_S(S/\mathfrak{p}S)$ for every $\mathfrak{p} \in \mathrm{Assh}(R)$.

Further, if $R$ is a homomorphic image of a Cohen–Macaulay local ring, then the next condition is also equivalent:

(c) $S$ is an approximately Cohen–Macaulay ring.

We will also prove the following result. Let $R$ and $S$ be nonzero $k$-algebras such that $T := R \otimes_k S$ is Noetherian. Assume that $R$ is not a Cohen–Macaulay ring. Then the following hold (see Theorem 10):

(i) If $R$ is an approximately Cohen–Macaulay ring and $S$ is a Cohen–Macaulay ring, then $T$ is an approximately Cohen–Macaulay ring;
(ii) If $T$ is an approximately Cohen–Macaulay ring, then $S$ is a Cohen–Macaulay ring;
(iii) If $R$ is a homomorphic image of a Cohen–Macaulay ring or $k$ is algebraically closed, then the following conditions are equivalent:
    (a) $T$ is an approximately Cohen–Macaulay ring;
    (b) $R$ is an approximately Cohen–Macaulay ring and $S$ is a Cohen–Macaulay ring.

## 2. MAIN RESULTS

For a finitely generated $R$-module $M$ of finite Krull dimension, recall that

$$\mathrm{Assh}_R(M) = \{\mathfrak{p} \in \mathrm{Supp}_R(M) \mid \dim(R/\mathfrak{p}) = \dim(\mathrm{M})\},$$

and denote $\mathrm{Assh}_R(R) = \mathrm{Assh}(R)$. Let $U_R(0) = \bigcap_{\mathfrak{p}\in \mathrm{Assh}(R)} I(\mathfrak{p})$, where $(0) = \bigcap_{\mathfrak{p}\in \mathrm{Ass}(R)} I(\mathfrak{p})$ denotes a minimal primary decomposition of the zero ideal of $R$ (cf. Goto, 1982).

Let $M$ be a finitely generated $R$-module and $I$ an ideal of $R$ such that $IM \neq M$. Then the common length of the maximal $M$-sequences in $I$ is called the *grade* of $I$ on $M$, denoted by $\mathrm{grade}_M(I)$. If $(R, \mathfrak{m})$ is a local ring, and $M$ is a finitely generated nonzero $R$-module, then the grade of $\mathfrak{m}$ on $M$ is called the *depth* of $M$, denoted by $\mathrm{depth}(M)$.

**Theorem 1** (see Goto, 1982). *Let $R$ be a local ring with maximal ideal $\mathfrak{m}$ and $\dim(R) = d$. Suppose that $R$ is not a Cohen–Macaulay ring. Then the following conditions are equivalent*:

(i) *$R$ is an approximately Cohen–Macaulay ring*;

(ii) *$R$ contains an ideal $I$ such that $I$ is a Cohen–Macaulay $R$-module of dimension $d-1$ and $R/I$ is a Cohen–Macaulay ring of dimension $d$*;

(iii) *$R/U_R(0)$ is a Cohen–Macaulay ring and* $\mathrm{depth}(R) = d - 1$;

(iv) (a) *$H^i_{\mathfrak{m}}(R) = (0)$ for $i \neq d-1, d$*;

  (b) *$\mathrm{Hom}_R(H^{d-1}_{\mathfrak{m}}(R), E_R(R/\mathfrak{m}))$ is a Cohen–Macaulay $\widehat{R}$-module of dimension $d-1$*;

  (c) *The local ring $R/\mathfrak{p}$ is unmixed for every $\mathfrak{p} \in \mathrm{Assh}(R)$, i.e., the equality $\dim(\widehat{R}/P) = d$ holds for every $P \in \mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ and for every $\mathfrak{p} \in \mathrm{Assh}(R)$.*

*In this case, the ideal $I$ appearing in assertion* (ii) *is uniquely determined and equals $U_R(0)$. Here $\widehat{R}$* (*resp.* $E_R(R/\mathfrak{m})$) *denotes the $\mathfrak{m}$-adic completion of $R$* (*resp. the injective hull of $R/\mathfrak{m}$*).

In the following theorem we consider the behavior of approximately Cohen–Macaulay property by passing to localizations.

**Theorem 2.** *Let $(R, \mathfrak{m})$ be an approximately Cohen–Macaulay ring. Then*:

(i) *For any $\mathfrak{p} \in \mathrm{Spec}(R)$, $\dim(R_{\mathfrak{p}}) - \mathrm{depth}(R_{\mathfrak{p}}) \leq 1$.*

(ii) *Suppose $R$ is not a Cohen–Macaulay ring. Then for any $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $R_{\mathfrak{p}}$ is not a Cohen–Macaulay ring, $\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$.*

(iii) *Suppose $R$ is not a Cohen–Macaulay ring. Then for any $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $R_{\mathfrak{p}}$ is not a Cohen–Macaulay ring, $U_{R_{\mathfrak{p}}}(0) = U_R(0)R_{\mathfrak{p}}$.*

(iv) *For any $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_{\mathfrak{p}}$ is an approximately Cohen–Macaulay ring.*

*Proof.* (i) This follows from the fact that $\dim(R_{\mathfrak{p}}) - \mathrm{depth}(R_{\mathfrak{p}}) \leq \dim(R) - \mathrm{depth}(R)$ for any $\mathfrak{p} \in \mathrm{Spec}(R)$ (see Matsumura, 1986, Exercise 17.5(ii)).

(ii) Let $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $R_{\mathfrak{p}}$ is not a Cohen–Macaulay ring. By (i), we have $\mathrm{depth}(R_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}) - 1$. Also, in view of Matsumura (1986, Exercise 17.5(i)), we have

$$\begin{aligned} \mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) - 1 &\leq \dim(R) - 1 = \mathrm{depth}(R) \\ &\leq \mathrm{grade}_R(\mathfrak{p}) + \dim(R/\mathfrak{p}) \\ &\leq \mathrm{depth}(R_{\mathfrak{p}}) + \dim(R/\mathfrak{p}). \end{aligned}$$

Therefore, $\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$.

(iii) Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $R_{\mathfrak{p}}$ is not a Cohen–Macaulay ring. We claim that $U_R(0)R_{\mathfrak{p}} = U_{R_{\mathfrak{p}}}(0)$. If $(0) = \bigcap_{\mathfrak{q}\in \operatorname{Ass}(R)} I(\mathfrak{q})$ is an irredundant primary decomposition for the zero ideal of $R$, then

$$(0) = \bigcap_{\substack{\mathfrak{q}\in \operatorname{Ass}(R)\\ \mathfrak{q}\subseteq \mathfrak{p}}} I(\mathfrak{q})R_{\mathfrak{p}}$$

is a minimal primary decomposition for the zero ideal of $R_{\mathfrak{p}}$. Thus it is enough to show that

$$\operatorname{Assh}(R_{\mathfrak{p}}) = \{\mathfrak{q}R_{\mathfrak{p}} \mid \mathfrak{q} \in \operatorname{Assh}(R), \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Let $\mathfrak{q} \in \operatorname{Spec}(R)$. We have

$$\begin{aligned}\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Assh}(R_{\mathfrak{p}}) &\Longleftrightarrow \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \dim(R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}})\\ &\Longleftrightarrow \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) = \operatorname{ht}(\mathfrak{p})\\ &\Longleftrightarrow \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim(R/\mathfrak{p}) = \dim(R).\end{aligned}$$

Let $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Assh}(R_{\mathfrak{p}})$. Since

$$\dim(R/\mathfrak{q}) \geq \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim(R/\mathfrak{p}) = \dim(R),$$

$\dim(R/\mathfrak{q}) = \dim(R)$ and hence $\mathfrak{q} \in \operatorname{Assh}(R)$.

Now, let $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \operatorname{Assh}(R)$. Since $R/U_R(0)$ is a catenary ring, $R/\mathfrak{q}$ is also catenary and hence by Matsumura (1986, Theorem 31.4), $\operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim(R/\mathfrak{p}) = \dim(R/\mathfrak{q})$. Thus $\operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim(R/\mathfrak{p}) = \dim(R)$. Therefore, $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Assh}(R_{\mathfrak{p}})$.

(iv) Let $\mathfrak{p} \in \operatorname{Spec}(R)$. If $R_{\mathfrak{p}}$ is a Cohen–Macaulay ring, then $R_{\mathfrak{p}}$ is an approximately Cohen–Macaulay ring. If $R_{\mathfrak{p}}$ is not a Cohen–Macaulay ring, then $R$ is not a Cohen–Macaulay ring, and so $R/U_R(0)$ is a Cohen–Macaulay ring. Now, by (iii), $R_{\mathfrak{p}}/U_{R_{\mathfrak{p}}}(0)$ is Cohen–Macaulay. Thus the assertion follows from (i) and Theorem 1(iii). □

By using Theorem 2, the concept of approximately Cohen–Macaulay ring can be extended to nonlocal rings by defining that a ring $R$ is approximately Cohen–Macaulay if for all prime ideals $\mathfrak{p}$ of $R$, and the ring $R_{\mathfrak{p}}$ is an approximately Cohen–Macaulay ring.

In the following result we consider the behavior of $\operatorname{Assh}(-)$ and the primary decomposition of the zero submodule under base change.

**Proposition 3.** *Let $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a flat local homomorphism. Then the following hold*:

(i) $\operatorname{Assh}(R) = \{\mathfrak{q} \cap R \mid \mathfrak{q} \in \operatorname{Assh}(S)\}$;
(ii) $U_R(0)S \subseteq U_S(0)$;
(iii) *The following conditions are equivalent*:

(a) $U_R(0)S = U_S(0)$;

(b) $\operatorname{Assh}(S) = \{\mathfrak{q} \in \operatorname{Ass}(S) \mid \mathfrak{q} \cap R \in \operatorname{Assh}(R)\}$;
(c) $\operatorname{Ass}_S(S/\mathfrak{p}S) = \operatorname{Assh}_S(S/\mathfrak{p}S)$ *for every* $\mathfrak{p} \in \operatorname{Assh}(R)$.

*Proof.* (i) Let $\mathfrak{p} \in \operatorname{Assh}(R)$. Then $\dim(R) = \dim(R/\mathfrak{p})$. Consider the flat local homomorphism $\tilde{\varphi} : R/\mathfrak{p} \to S/\mathfrak{p}S$. Hence,

$$\dim(R) = \dim(S/\mathfrak{p}S) - \dim(S/\mathfrak{m}S),$$

and therefore $\dim(S) = \dim(S/\mathfrak{p}S)$. So there exists a minimal prime ideal of $\mathfrak{p}S$, say $\mathfrak{q}$, such that $\dim(S) = \dim(S/\mathfrak{q})$. Since $\mathfrak{q}$ is a minimal prime ideal of $\mathfrak{p}S$, $\mathfrak{q} \cap R = \mathfrak{p}$.

Let $\mathfrak{q} \in \operatorname{Assh}(S)$ and $\mathfrak{q} \cap R = \mathfrak{p}$. We have $\dim(S/\mathfrak{q}) = \dim(S)$ and $\mathfrak{p}S \subseteq \mathfrak{q}$; that means $\dim(S/\mathfrak{p}S) = \dim(S)$. Thus,

$$\dim(R/\mathfrak{p}) + \dim(S/\mathfrak{m}S) = \dim(S),$$

and so $\dim(R/\mathfrak{p}) = \dim(R)$.

(ii) It is enough to show that $U_R(0)S_\mathfrak{q} = (0)S_\mathfrak{q}$ for every $\mathfrak{q} \in \operatorname{Assh}(S)$. Indeed, since $S$ is flat,

$$U_R(0)S_\mathfrak{q} = \bigcap_{\mathfrak{p}\in\operatorname{Assh}(R)} I(\mathfrak{p})S_\mathfrak{q} = \bigcap_{\substack{\mathfrak{p}\in\operatorname{Assh}(R)\\ \mathfrak{p}\subseteq\mathfrak{q}\cap R}} I(\mathfrak{p})S_\mathfrak{q}.$$

If $\mathfrak{q} \in \operatorname{Assh}(S)$, then $\mathfrak{q} \cap R = \mathfrak{p}$. We know that $(I(\mathfrak{p})R_\mathfrak{p})S_\mathfrak{q}$ and $(0)S_\mathfrak{q}$ are $\mathfrak{p}R_\mathfrak{p}$-primary submodules of the $R_\mathfrak{p}$-module $S_\mathfrak{q}$. Thus $I(\mathfrak{p})S_\mathfrak{q} = (I(\mathfrak{p})R_\mathfrak{p})S_\mathfrak{q} = (0)S_\mathfrak{q}$, as required.

(iii) It is known that (cf. Matsumura, 1986, Theorem 23.2)

$$\{\mathfrak{p}\} = \{\varphi^{-1}(\mathfrak{q}) \mid \mathfrak{q} \in \operatorname{Ass}_S(S/\mathfrak{p}S)\} \qquad \text{for each } \mathfrak{p} \in \operatorname{Spec}(R),$$

$$\operatorname{Ass}(S) = \bigcup_{\mathfrak{p}\in\operatorname{Ass}(R)} \operatorname{Ass}_S(S/\mathfrak{p}S),$$

$$\operatorname{Ass}_S(S/U_R(0)S) = \bigcup_{\mathfrak{p}\in\operatorname{Assh}(R)} \operatorname{Ass}_S(S/\mathfrak{p}S),$$

$$\operatorname{Assh}(S) = \operatorname{Ass}_S(S/U_S(0)) = \bigcup_{\mathfrak{p}\in\operatorname{Assh}(R)} \operatorname{Assh}_S(S/\mathfrak{p}S).$$

Also note that $\bigcap_{\mathfrak{p}\in\operatorname{Ass}(R)} I(\mathfrak{p})S = (0)$ is a minimal primary decomposition of the zero submodule of the $R$-module $S$ and if $\bigcap_{1\leq i\leq n} Q_i = (0)$ is a minimal primary decomposition of the zero ideal in $S$, then $\bigcap_{1\leq i\leq n} Q_i = (0)$ is a primary decomposition of the zero submodule of the $R$-module $S$. We can now easily obtain (iii). □

The next result shows that the approximately Cohen–Macaulay property is stable under specialization.

**Lemma 4.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring with* $\dim(R) = d$. *Let* $x \in \mathfrak{m}\backslash Z(R)$. *If* $R$ *is an approximately Cohen–Macaulay ring, then* $R/xR$ *is an approximately Cohen–Macaulay ring.*

*Proof.* We may assume that $R$ is not a Cohen–Macaulay ring. Then there exists an ideal $I$ of $R$ such that $I$ is a Cohen–Macaulay $R$-module, $\dim(I) = d-1$, and $R/I$ is a Cohen–Macaulay ring of dimension $d$. Therefore, $I/xI$ is a Cohen–Macaulay $R/xR$-module of dimension $d-2$. Since $\mathrm{Ass}_R(R/I) = \{\mathfrak{q} \in \mathrm{Assh}(R) \mid I \subseteq \mathfrak{q}\}$, $x \notin Z_R(R/I)$ and hence $R/(I+xR)$ is a Cohen–Macaulay ring of dimension $d-1$ and $I \cap xR = xI$. By using the isomorphism $(I+xR)/xR \cong I/(I \cap xR)$, we obtain that $(I+xR)/xR$ is a Cohen–Macaulay $R/xR$-module of dimension $d-2$. The assertion now follows from Theorem 1. □

**Lemma 5.** *Let $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a flat local homomorphism. Let $S$ be an approximately Cohen–Macaulay ring. Then either $R$ or $S/\mathfrak{m}S$ is Cohen–Macaulay.*

*Proof.* Assume that $R$ is not Cohen–Macaulay. Then we have

$$\begin{aligned}\mathrm{depth}(R) &= \mathrm{depth}(S) - \mathrm{depth}(S/\mathfrak{m}S) = \dim(S) - 1 - \mathrm{depth}(S/\mathfrak{m}S)\\ &\geq \dim(S) - 1 - \dim(S/\mathfrak{m}S) = \dim(R) - 1.\end{aligned}$$

Since $R$ is not Cohen–Macaulay, $\mathrm{depth}(R) = \dim(R) - 1$ and hence $S/\mathfrak{m}S$ is Cohen–Macaulay. □

We are now ready to prove that the approximately Cohen–Macaulay property is stable (in some sense) under change of ring. This result is somehow parallel to the results on properties like regular, complete intersection and Cohen–Macaulay (cf. Tousi and Yassemi, 2003, Theorem 1).

**Theorem 6.** *Let $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a flat local homomorphism. Assume that $R$ is not a Cohen–Macaulay ring. Then the following are equivalent*:

(a) *$R$ is an approximately Cohen–Macaulay ring and $S/\mathfrak{m}S$ is a Cohen–Macaulay ring*;
(b) *$S$ is an approximately Cohen–Macaulay ring and $\mathrm{Ass}_S(S/\mathfrak{p}S) = \mathrm{Assh}_S(S/\mathfrak{p}S)$ for every $\mathfrak{p} \in \mathrm{Assh}(R)$.*

*Further, if $R$ is a homomorphic image of a Cohen–Macaulay local ring, then the next condition is also equivalent*:

(c) *$S$ is an approximately Cohen–Macaulay ring.*

*Proof.* Consider the induced flat local homomorphism $\tilde{\varphi} : R/U_R(0) \longrightarrow S/U_R(0)S$.

(a) $\Longrightarrow$ (b) By Bruns and Herzog (1993, Theorem 2.1.7) $S/U_R(0)S$ is a Cohen–Macaulay ring. Also the following (in)equalities hold by Proposition 3(ii):

$$\dim(S) \geq \dim(S/U_R(0)S) \geq \dim(S/U_S(0)) = \dim(S).$$

On the other hand, $U_R(0)$ is a Cohen–Macaulay $R$-module of dimension $\dim(R)-1$. Thus $U_R(0)S \cong U_R(0) \otimes_R S$ is a Cohen–Macaulay $S$-module of dimension $\dim(S)-1$, because

$$\begin{aligned}\dim(S) - 1 &= \dim(R) - 1 + \dim(S/\mathfrak{m}S) = \dim(U_R(0)) + \dim(S/\mathfrak{m}S)\\ &= \dim(U_R(0) \otimes_R S).\end{aligned}$$

The last paragraph of Theorem 1 implies that $U_R(0)S = U_S(0)$. Now, the assertions follow from Proposition 3 and Theorem 1(ii).

(b) $\Longrightarrow$ (a) By Lemma 5, $S/\mathfrak{m}S$ is a Cohen–Macaulay ring. We have $U_S(0) = U_R(0)S$ by Proposition 3(iii). On the other hand, $S$ is not a Cohen–Macaulay ring and so $S/U_R(0)S$ is a Cohen–Macaulay ring. Therefore, $R/U_R(0)$ is a Cohen–Macaulay ring. Now, the assertion follows from the following equalities:

$$\dim(R) = \dim(S) - \dim(S/\mathfrak{m}S) = \operatorname{depth}(S) + 1 - \operatorname{depth}(S/\mathfrak{m}S) = \operatorname{depth}(R) + 1.$$

(b) $\Longrightarrow$ (c) It is clear.

(c) $\Longrightarrow$ (a) By Lemma 5, we know that $S/\mathfrak{m}S$ is Cohen–Macaulay. Assume that $\mathfrak{q}$ is a minimal ideal of $\mathrm{V}(\mathfrak{m}S)$. Then $\mathfrak{q} \cap R = \mathfrak{m}$ and $\dim(S_{\mathfrak{q}}/\mathfrak{m}S_{\mathfrak{q}}) = 0$. By considering the induced homomorphism $\hat{\varphi} : R \longrightarrow S_{\mathfrak{q}}$ one can reduce to the case where $\dim(S/\mathfrak{m}S) = 0$. By using "(a) $\Longrightarrow$ (b)" and Goto (1982, Corollary 2.6), we may assume that $R$ and $S$ are complete. Note that $S$ is not a Cohen–Macaulay ring. We use induction on $\dim(S) = n$. If $n = 1$, then $\dim(R) + \dim(S/\mathfrak{m}S) = 1$ and hence $\dim(R) = 1$. Thus, $R$ is an approximately Cohen–Macaulay ring. Now suppose, inductively, that $n \geq 2$ and we have established the result for $n - 1$. Set $N = \operatorname{Hom}_R(\mathrm{H}^{n-1}_{\mathfrak{m}}(R), \mathrm{E}_R(R/\mathfrak{m}))$. Since

$$\operatorname{depth}(R) = \operatorname{depth}(S) = \dim(S) - 1 = \dim(R) - 1,$$

$\mathrm{H}^{i}_{\mathfrak{m}}(R) = (0)$ for $i \notin \{n-1, n\}$. Therefore, it is enough to show that $N$ is a Cohen–Macaulay $R$-module of dimension $n - 1$.

We claim that $\mathfrak{m} \notin \operatorname{Ass}_R(N)$. Otherwise $\mathfrak{m} \in \operatorname{Att}_R(\mathrm{H}^{n-1}_{\mathfrak{m}}(R))$ and so by Brodmann and Sharp (1998, Exercise 11.3.7), $\mathfrak{n} \in \operatorname{Att}_S(\mathrm{H}^{n-1}_{\mathfrak{n}}(S))$. Therefore, $\mathfrak{n} \in \operatorname{Ass}_S(\operatorname{Hom}_S(\mathrm{H}^{n-1}_{\mathfrak{n}}(S), \mathrm{E}_S(S/\mathfrak{n})))$. But $\operatorname{Hom}_S(\mathrm{H}^{n-1}_{\mathfrak{n}}(S), \mathrm{E}_S(S/\mathfrak{n}))$ is a Cohen–Macaulay $S$-module of dimension $n - 1 \geq 1$. That is a contradiction.

Since $N$ is a finitely generated $R$-module, the set $\operatorname{Ass}_R(N)$ is finite and hence there exists $x \in \mathfrak{m} \setminus (Z(R) \cup Z_R(N))$. Consider the induced flat local homomorphism $\hat{\varphi} : R/xR \longrightarrow S/\varphi(x)S$. Since $\varphi(x) \notin Z(S)$, by Lemma 4, $S/\varphi(x)S$ is an approximately Cohen–Macaulay ring of dimension $n - 1$. Therefore, by the inductive hypothesis $R/xR$ is an approximately Cohen–Macaulay module of dimension $n - 1$. Set $\overline{R} = R/xR$ and $\bar{\mathfrak{m}} = \mathfrak{m}/xR$. The $R/xR$-module $\operatorname{Hom}_{\overline{R}}(\mathrm{H}^{n-2}_{\bar{\mathfrak{m}}}(\overline{R}), \mathrm{E}_{\overline{R}}(\overline{R}/\bar{\mathfrak{m}}))$ is a Cohen–Macaulay module of dimension $n - 2$. The exact sequence

$$0 \to R \xrightarrow{x.} R \to R/xR \to 0$$

induces the exact sequence

$$0 \to \mathrm{H}^{n-2}_{\mathfrak{m}}(R/xR) \to \mathrm{H}^{n-1}_{\mathfrak{m}}(R) \xrightarrow{x.} \mathrm{H}^{n-1}_{\mathfrak{m}}(R).$$

Therefore, $\mathrm{H}^{n-2}_{\bar{\mathfrak{m}}}(\overline{R}) \cong \operatorname{Hom}_R(R/xR, \mathrm{H}^{n-1}_{\mathfrak{m}}(R))$. By Brodmann and Sharp (1998, Lemma 10.1.15),

$$\operatorname{Hom}_{\overline{R}}(\mathrm{H}^{n-2}_{\bar{\mathfrak{m}}}(\overline{R}), \mathrm{E}_{\overline{R}}(\overline{R}/\bar{\mathfrak{m}})) \cong \operatorname{Hom}_{\overline{R}}(\operatorname{Hom}_R(\overline{R}, \mathrm{H}^{n-1}_{\mathfrak{m}}(R)), \operatorname{Hom}_R(\overline{R}, \mathrm{E}_R(R/\mathfrak{m})))$$

$$\begin{aligned}&\cong \mathrm{Hom}_R(\mathrm{Hom}_R(\overline{R}, \mathrm{H}^{n-1}_{\mathfrak{m}}(R)), \mathrm{E}_R(R/\mathfrak{m}))\\ &\cong \overline{R} \otimes_R \mathrm{Hom}_R(\mathrm{H}^{n-1}_{\mathfrak{m}}(R), \mathrm{E}_R(R/\mathfrak{m})).\end{aligned}$$

Thus $N/xN$ is a Cohen–Macaulay $R/xR$-module of dimension $n-2$. Since $x \notin Z_R(N)$, $N$ is a Cohen–Macaulay $R$-module of dimension $n-1$. □

Note that in Theorem 6, the condition "$\mathrm{Ass}_S(S/\mathfrak{p}S) = \mathrm{Assh}_S(S/\mathfrak{p}S)$ for every $\mathfrak{p} \in \mathrm{Assh}(R)$" is not superficial, as the following example shows.

**Example 7** (See Nagata, 1962). Let $(R, \mathfrak{m})$ be a 2-dimensional local domain for which the $\mathfrak{m}$-adic completion $\widehat{R} \cong k[[x, y, z]]/(xy, xz)$. Put $S = \widehat{R}$. Let $\varphi : R \longrightarrow S$ be a natural ring homomorphism. Then:

(i) $R$ is not approximately Cohen–Macaulay local domain. In particular, $\mathrm{Assh}(R) = \{(0)\}$;
(ii) $S/\mathfrak{m}S = k$ is regular, and thus is Cohen–Macaulay;
(iii) $S$ is approximately Cohen–Macaulay, but not unmixed;
(iv) $R$ is not a homomorphic image of a Cohen–Macaulay local ring.

**Corollary 8.** *Let $\varphi : R \longrightarrow S$ be a flat homomorphism. If $R$ is an approximately Cohen–Macaulay ring and $(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \otimes_R S$ is a Cohen–Macaulay ring for every* $\mathfrak{p} \in \mathrm{Spec}(R)$, *then $S$ is an approximately Cohen–Macaulay ring.*

*Proof.* Let $\mathfrak{q} \in \mathrm{Spec}(S)$. Set $\mathfrak{p} = \mathfrak{q} \cap R \in \mathrm{Spec}(R)$. The induced homomorphism $\tilde{\varphi} : R_\mathfrak{p} \longrightarrow S_\mathfrak{q}$ is a flat local homomorphism. It is clear that $S_\mathfrak{q}/(\mathfrak{p}R_\mathfrak{p})S_\mathfrak{q}$ is a localization of $(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \otimes_R S$. Now, the assertion follows from Theorem 6. □

**Corollary 9.** *Let $\varphi : R \longrightarrow S$ is a faithfully flat homomorphism. Suppose that $R$ is not a Cohen–Macaulay ring, but a homomorphic image of a Cohen–Macaulay ring. If $S$ is an approximately Cohen–Macaulay ring, then $R$ is an approximately Cohen–Macaulay ring.*

*Proof.* Assume that $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\mathfrak{q}$ is a minimal ideal of $\mathrm{V}(\mathfrak{p}S)$. Then $\mathfrak{q} \cap R = \mathfrak{p}$ and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$. Consider the induced homomorphism $\hat{\varphi} : R_\mathfrak{p} \longrightarrow S_\mathfrak{q}$. The assertion now follows from Theorem 6. □

**Theorem 10.** *Let $R$ and $S$ be nonzero $k$-algebras such that $T := R \otimes_k S$ is Noetherian. Assume that $R$ is not a Cohen–Macaulay ring. Then the following hold*:

(i) *If $R$ is an approximately Cohen–Macaulay ring and $S$ is a Cohen–Macaulay ring, then $T$ is an approximately Cohen–Macaulay ring*;
(ii) *If $T$ is an approximately Cohen–Macaulay ring, then $S$ is a Cohen–Macaulay ring*;
(iii) *If $R$ is a homomorphic image of a Cohen–Macaulay ring or $k$ is algebraically closed, then the following conditions are equivalent*:

  (a) *$T$ is an approximately Cohen–Macaulay ring*;
  (b) *$R$ is an approximately Cohen–Macaulay ring and $S$ is a Cohen–Macaulay ring.*

*Proof.* (i) Consider the faithfully flat homomorphism $\varphi : R \longrightarrow (R \otimes_k S)$. It is enough to show that the fibers $(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \otimes_R (R \otimes_k S) \cong (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \otimes_k S$ over every prime ideal $\mathfrak{p}$ of $R$ are Cohen–Macaulay rings. Since $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a Cohen–Macaulay ring (it is actually a field), $(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \otimes_k S$ is also a Cohen–Macaulay ring by Tousi and Yassemi (2003, Theorem 6).

(ii) Assume that $S$ is not a Cohen–Macaulay ring. Then there exist $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\mathfrak{q} \in \operatorname{Spec}(S)$ such that $R_\mathfrak{p}$ and $S_\mathfrak{q}$ are not Cohen–Macaulay rings, and hence $\operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) \leq \operatorname{ht}(\mathfrak{p}) - 1$ and $\operatorname{grade}_{S_\mathfrak{q}}(\mathfrak{q}S_\mathfrak{q}) \leq \operatorname{ht}(\mathfrak{q}) - 1$. Therefore,

$$\operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) + \operatorname{grade}_{S_\mathfrak{q}}(\mathfrak{q}S_\mathfrak{q}) \leq \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) - 2. \qquad (*)$$

There exists $Q \in \operatorname{Spec}(R \otimes_k S)$ such that $Q \cap R = \mathfrak{p}$ and $Q \cap S = \mathfrak{q}$. On the other hand, by Bouchiba and Kabbaj (2002, Proposition 2.3),

$$\operatorname{ht}(Q) = \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) + \operatorname{ht}(Q/((\mathfrak{p} \otimes_k S) + (R \otimes_k \mathfrak{q})))$$

and

$$\begin{aligned}\operatorname{grade}_{(R\otimes_k S)_Q}(Q(R \otimes_k S)_Q) = {} & \operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) + \operatorname{grade}_{S_\mathfrak{q}}(\mathfrak{q}S_\mathfrak{q}) \\ & + \operatorname{ht}(Q/((\mathfrak{p} \otimes_k S) + (R \otimes_k \mathfrak{q}))).\end{aligned}$$

But

$$\operatorname{grade}_{(R\otimes_k S)_Q}(Q(R \otimes_k S)_Q) \geq \operatorname{ht}(Q) - 1,$$

so we have

$$\operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) + \operatorname{grade}_{S_\mathfrak{q}}(\mathfrak{q}S_\mathfrak{q}) \geq \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) - 1$$

and by using $(*)$, $\operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) - 1 \leq \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) - 2$. That is a contradiction.

(iii) In fact, (b) $\Longrightarrow$ (a) is just (i), and for proving (a) $\Longrightarrow$ (b); by (ii), it is enough to show that $R$ is an approximately Cohen–Macaulay ring.

First, let $R$ be a homomorphic image of a Cohen–Macaulay ring. Consider the faithfully flat homomorphism $\varphi : R \longrightarrow (R \otimes_k S)$. The assertion now follows from Corollary 9.

Next, let $k$ be algebraically closed. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then there exists $\mathfrak{q} \in \operatorname{Spec}(R \otimes_k S)$ with $\mathfrak{q} \cap R = \mathfrak{p}$. Consider the induced flat local homomorphism $\tilde{\varphi} : R_\mathfrak{p} \longrightarrow (R \otimes_k S)_\mathfrak{q}$. Let $\mathfrak{q}' \in \operatorname{Ass}(R \otimes_k S)$ with $\mathfrak{q}' \subseteq \mathfrak{q}$ and $\operatorname{ht}(\mathfrak{p}/(\mathfrak{q}' \cap R)) = \operatorname{ht}(\mathfrak{p})$. By using Proposition 3 and Theorem 6, it is enough to show that $\operatorname{ht}(\mathfrak{q}/\mathfrak{q}') = \operatorname{ht}(\mathfrak{q})$. Set $\mathfrak{q}' \cap R = \mathfrak{p}'$, $\mathfrak{q} \cap S = \mathfrak{p}_2$, and $\mathfrak{q}' \cap S = \mathfrak{p}_1$. Then by Matsumura (1986, Theorem 23.2), $\mathfrak{q}' \in \operatorname{Ass}_T(T/\mathfrak{p}'T)$. On the other hand, $S$ is Cohen–Macaulay and hence by Tousi and Yassemi (2003, Theorem 6), $T/\mathfrak{p}'T = (R/\mathfrak{p}') \otimes_k S$ satisfies Serre's condition $(S_1)$. Therefore, $\mathfrak{q}' \in \operatorname{Min}(\mathfrak{p}'T)$ and so $\mathfrak{q}' \in \operatorname{Min}(\mathfrak{p}'T + \mathfrak{p}_1 T)$. On the other hand, $T/(\mathfrak{p}'T +$

$\mathfrak{p}_1T) \cong (R/\mathfrak{p}') \otimes_k (S/\mathfrak{p}_1)$ is an integral domain (see Eisenbud, 1995, Exercise A1.2(a), p. 562). Thus $\mathfrak{q}' = \mathfrak{p}'T + \mathfrak{p}_1T$. Now, the following equalities hold:

$$\begin{aligned}\mathrm{ht}(\mathfrak{q}/\mathfrak{q}') &= \mathrm{ht}(\mathfrak{q}/(\mathfrak{p}'T + \mathfrak{p}_1T)) = \mathrm{ht}(\mathfrak{p}/\mathfrak{p}') + \mathrm{ht}(\mathfrak{p}_2/\mathfrak{p}_1) + \mathrm{ht}(\mathfrak{q}/(\mathfrak{p}T + \mathfrak{p}_2T))\\ &= \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}_2) + \mathrm{ht}(\mathfrak{q}/(\mathfrak{p}T + \mathfrak{p}_2T)) = \mathrm{ht}(\mathfrak{q}),\end{aligned}$$

where the second and the last equalities hold by Bouchiba and Kabbaj (2002, Proposition 2.3), and the third one uses the fact that $\mathfrak{p}_1 \in \mathrm{Ass}(S)$. □

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